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Renormalization Group Study of Quantum Fluctuations near Classical Critical Points of Hamiltonian Systems

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A general framework is considered for treating quantum corrections to the classical limit in the Wigner function formalism. We discuss the quantal effect on the classical phenomena such as period doubling and the breakup of KAM tori. By using an exact renormalization group method, the scaling factor for Planck's constant is derived as an eigenvalue of the linearized renormalization transformation.

KEY WORDS: Renormalization group; Hamiltonian; dynamical phase transition; Planck's constant scaling; period doubling; last KAM torus; critical exponents.

1. INTRODUCTION

Since the discovery by Feigenbaum that near the accumulation point the period-doubling sequence possesses universal properties over a class of dissipative dynamical systems,[(1)] a multitude of nonlinear phenomena have been found to exhibit critical behavior with universal characteristics. Such are in dissipative systems the intermittent and mode-locking routes to chaos as well as the effect of quasiperiodic and random perturbations on different routes,[4] and universal properties have been also detected in collective modes of coupled maps and oscillators.[(4)] Further examples arise in Hamiltonian systems, as the period n-tupling sequences and the breakup of

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[4] Most of these phenomena are reviewed and extensive references can be found in ref. 2. For quasiperiodic perturbations see ref. 3.

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the last Kolmogorov–Arnold–Moser (KAM) torus in area-preserving maps(see refs. 5 for reviews), transients therein caused by random forces,[6] and diverse scenarios in higher-dimensional symplectic maps.[7] Finally, we mention the crossover from dissipative to Hamiltonian behavior in a large family of chaotic systems, where a universal series of dissipation parameter values characterizes the sequential disappearance of strange attractors.[8]

The behavior in nonlinear dynamical systems exactly corresponds to the behavior of thermodynamic systems near critical points. The critical point occurs at a definite value of some system parameter, for example, there is a critical temperature or a critical strength of nonlinearity parameter. In the thermodynamic limit, i.e., the limit of large system size, characterized by a length L becoming infinite, singularities appear in the thermodynamic functions as the temperature approaches the critical temperature. These singularities are a result of the divergence of a characteristic coherence length ξ as criticality is approached. The singularities are expressed as universal critical exponents and universal amplitude ratios and a universal scaling function. For finite but large L, one can make precise statements using the ideas of finite-size scaling. The term universal means that the same exponents, etc., are found for a large class of systems. Usually, one must change some symmetry of the system in order to change the exponents.

The parameter in nonlinear dynamical systems corresponding to system size L is the time. For example, Feigenbaum found a transition to chaos at the limit of the period-doubling sequence of the logistic map, i.e., for infinitely many iterations of the map. Scaling functions and critical exponents are also found.

The simplest situation is one in which only one parameter needs to be tuned to criticality. All the other parameters are "irrelevant." However, it is usual to find other parameters which must also be made critical, for example, the magnetic field in the case of a ferromagnetic phase transition. Typically these "relevant" parameters break a symmetry of the system (when they become nonzero), that is, taking into account a relevant parameter enlarges the class of systems in a significant way. Corresponding to a relevant parameter is an exponent describing the system's behavior for small values of the parameter. Generally, this exponent cannot be trivially found from the consideration of the restricted class of systems for which the parameter is zero.

Aside from dimensionality, the most typical relevant parameters for dynamical systems are those giving the strength of dissipation or of noise. Dissipation, if present, destroys the phase-space volume-preserving character of Hamiltonian systems, and obviously affects the long-time behavior in a dramatic way, thus indicating that it is relevant. Similarly, noise is relevant, and there are generally critical exponents showing how the system properties scale with small but finite dissipation or noise.

This article considers the relevant parameter $\hbar$, Planck's constant, the parameter representing the generalization of classical dynamics to quantum dynamics. It has previously been shown that $\hbar$ is indeed relevant and the corresponding critical exponent has been found.[9] However, this critical exponent is apparently a trivial concatenation of the classical exponent for scaling in the momentum direction with that for scaling in coordinate direction. On the other hand, the Feynman path integral formulation of quantum mechanics suggests that quantum effects are due to fluctuating, i.e., noisy, classical paths, albeit paths which must be added up taking into account phase relations. Therefore, the question addressed in this paper is: Why does quantum noise result in a "trivial" quantum exponent, while classical noise has a nontrivial exponent?

2. RENORMALIZATION GROUP

The universal nature of critical phenomena is, in each case, based upon the properties of the renormalization group (RNG) transformation associated with the specific problem. In the generic situation, the RNG is defined on the space of dynamical systems under study (e.g., in the case of the Hamiltonian period-doubling scenario a set of area-preserving maps), and it transforms a system by a given prescription (e.g., iteration of the map and scaling) into another one. The RNG transformation thus forms a mapping in system space. The dynamical system in a critical state (e.g., an area-preserving map at the accumulation point of the period-doubling sequence) corresponds typically to a fixed point of the RNG transformation, whose unstable manifold represents the route(s) through which the critical state is reached by changing the control parameter(s) of the dynamical system, and whose stable manifold forms the subspace of systems at criticality. The elements of the stable manifold are attracted to the fixed point upon the RNG transformation, whereas a system off criticality is driven toward the unstable manifold. These features constitute the source of universality. Each system in a critical state behaves asymptotically like the one at the fixed point, and near criticality the behavior of a system is determined by those forming the unstable manifold. In the simplest case the unstable manifold has dimension one. Each additional relevant variable increases the dimension of the unstable manifold by one. An important common characteristic of critical phenomena is scale invariance, i.e., near criticality similar behavior is observed if certain variables are appropriately rescaled. The scale factors are universal in the sense that they characterize asymptotically all systems driven by the RNG transformation

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toward the fixed point or the unstable manifold. In particular, the factor for a control parameter is obtained as the eigenvalue of the RNG transformation linearized about the fixed point.

The suggestion of Grempel *et al.*[9] that universal scaling is not only the property of classical dynamical systems, but of quantum as well, is in one sense surprising. Namely, quantum systems can certainly not be described merely as classical systems with one extra parameter: rather there are profound differences in the essence of the two theories. However, within the framework of the RNG, direct comparisons can be made. It is our purpose to provide these comparisons. Specifically, in the Hamiltonian period-doubling scenario and near the destruction of KAM tori, the quantum map exhibits scale invariance if, simultaneously with the classical operations, Planck's constant $\hbar$ is appropriately rescaled. The value of the scale factor was proposed to be $|\alpha\beta|$, where α and β are the classical scaling constants for the phase space coordinates, with different respective values for period doubling and the KAM problem. Subsequently the conjecture has been further substantiated by a study of the Wigner function of a periodically kicked one-dimensional particle, where the classical action was found to determine the lowest order quantum correction.[10] Independently, for the area-preserving period-doubling problem, Graham formulated an approximate RNG transformation for the moments of quantum fluctuations up to $O(\hbar^2)$.[11] Using uniform moments and quadratic interpolation for the fixed-point map, he obtained a dominant eigenvalue, associated with $\hbar^2$, which was numerically close to $\alpha^2\beta^2$. A recent computer study by Radons and Prange[12] demonstrated the scale invariance of the quantum standard map in the situation where the classical map exhibited a critical KAM torus with the golden mean winding number.

In this paper the lowest order quantum fluctuations are discussed in the Wigner function formalism for a periodically driven one-dimensional particle. We study systems whose classical counterparts are close to the period-doubling accumulation point or to the breakup of the last KAM torus. In order to establish the linearized RNG transformation for the fluctuations in both problems, the composition rule of the propagator is worked out for arbitrary (phase-space-dependent) cubic cumulants. The composition is found to possess a symmetry: If the constituent cumulants are derived from the classical action in a certain way, then the cumulants resulting from their composition will have the same relation with respect to the composed action. It follows that the linearized RNG transformation preserves the dependence of the fluctuations on the classical action for arbitrary number of RNG iteration steps. Thus the standard RNG picture does not apply, in which typically an initial perturbation is independent of the perturbed system, and where a spectral decomposition of the linearized

RNG transformation is necessary to determine the evolution of a system near the fixed point. There asymptotically the eigenvalues with modulus larger than one arise as relevant, becoming the universal scale factors for the given phenomenon. This happens, e.g., in the case of classical random perturbations. Nonetheless, an important aspect of RNG theories is preserved in the quantum problem, namely, the scale factor of Planck's constant emerges as an eigenvalue. If the classical action approaches a fixed point (in either the period-doubling or the KAM problem), we will show that the quantum moments will converge to the eigendirection of the linearized RNG transformation with eigenvalue $\alpha^2\beta^2$, which is associated with $\hbar^2$.

3. WIGNER FUNCTION PROPAGATION

Consider a particle in one dimension driven by a force periodic in time with period T. The classical action for the period $(0, T)$ is

$$S(x, x') = \int_0^T L(x(t), \dot{x}(t), t)\, dt, \qquad x(0) = x, \quad x(T) = x' \tag{1}$$

where $x(t)$ in the argument of the Lagrangian L is the classical orbit, assumed to be unique for fixed initial x and final x' points. The action generates the momenta as

$$p = -\partial_1 S(x, x'), \qquad p' = \partial_2 S(x, x') \tag{2}$$

where $\partial_i S$ denotes the partial derivative in terms of the ith argument. If Eq. (2) can be uniquely solved for x' and p' with given x and p, as is the case near both the period-doubling accumulation point and the last KAM torus, then it uniquely determines the Poincaré map of the system.

The use of the Wigner function for the characterization of the quantum state has several advantages.[13] The Wigner function determines the expectation values of functions of the position and momentum operators, and in the $\hbar \to 0$ limit it goes over to a classical probability density function in phase space. For increasing $\hbar$, its "widening" starting out from a classical Dirac delta probability density can be interpreted as the signature of quantum fluctuations. A further advantage is that it is real. In the pure state with wave function $\psi(x)$ the Wigner function assumes the form

$$W(x, p) = \int_{-\infty}^{+\infty} \frac{d\xi}{2\pi\hbar} \exp\left(\frac{ip\xi}{\hbar}\right) \psi^*\left(x + \frac{\xi}{2}\right) \psi\left(x - \frac{\xi}{2}\right) \tag{3}$$

Equivalently, we can consider its Fourier transform in the momentum variable p, which reads as

$$\omega(x, \zeta) = \psi^*\left(x + \frac{\hbar\zeta}{2}\right)\psi\left(x - \frac{\hbar\zeta}{2}\right) \tag{4}$$

In what follows the time evolution of the above function will be studied by using the technique of path integrals.[14] If the wave function evolves from $t=0$ to $t=T$ as

$$\psi_T(x') = \int_{-\infty}^{+\infty} dx\, U(x' \mid x)\, \psi_0(x) \tag{5}$$

then the time evolution of the transformed Wigner function (4) is evidently given by

$$\omega_T(x', \zeta') = \int_{-\infty}^{+\infty} dx\, d\zeta\, \kappa(x', \zeta' \mid x, \zeta)\, \omega_0(x, \zeta) \tag{6}$$

with

$$\kappa(x', \zeta' \mid x, \zeta) = \hbar U^*\left(x' + \frac{\hbar\zeta'}{2} \,\middle|\, x + \frac{\hbar\zeta}{2}\right) U\left(x' - \frac{\hbar\zeta'}{2} \,\middle|\, x - \frac{\hbar\zeta}{2}\right) \tag{7}$$

Note that the kernel U depends also on $\hbar$ as a parameter, thus κ will depend on $\hbar$ in a more complicated fashion than indicated explicitly on the right-hand side. It follows from Eq. (7) that the propagator κ satisfies

$$\kappa^*(x', \zeta' \mid x, \zeta) = \kappa(x', -\zeta' \mid x, -\zeta) \tag{8}$$

which is equivalent to the property that the propagator of the Wigner function is real. Hence simple symmetry relations can be obtained for the modulus and the phase of the propagator. In particular, if

$$\kappa(x', \zeta' \mid x, \zeta) = \rho(x, \zeta, x', \zeta') \exp[-i\phi(x, \zeta, x', \zeta')] \tag{9}$$

with ρ and ϕ real, then

$$\rho(x, \zeta, x', \zeta') = \rho(x, -\zeta, x', -\zeta') \tag{10}$$

$$\phi(x, \zeta, x', \zeta') = -\phi(x, -\zeta, x', -\zeta') \tag{11}$$

Although these functions are not necessarily analytic in $\hbar$, low-order quantum corrections are expected to be well described by the formal expansions

$$\rho(x,\zeta,x',\zeta')\approx\rho_0(x,x') + \hbar^2[\rho_0^{(2)}(x,x')+\zeta^2\rho_{20}(x,x') + 2\zeta\zeta'\rho_{11}(x,x')+\zeta'^2\rho_{02}(x,x')]+O(\hbar^4) \tag{12}$$

$$\phi(x,\zeta,x',\zeta')\approx\zeta\phi_{10}(x,x')+\zeta'\phi_{01}(x,x') + \hbar^2[\zeta\phi_{10}^{(2)}(x,x')+\zeta'\phi_{01}^{(2)}(x,x') + \zeta^3\phi_{30}(x,x')+3\zeta^2\zeta'\phi_{21}(x,x') + 3\zeta\zeta'^2\phi_{12}(x,x')+\zeta'^3\phi_{03}(x,x')]+O(\hbar^4) \tag{13}$$

It is noteworthy that Eqs. (9), (12), and (13) correspond to an Airy-type approximation combined with derivatives of the Dirac delta for the propagator of the Wigner function,[15] which one can see by performing Fourier transformation in the variables ζ and ζ'.

In the $\hbar\to 0$ limit, as opposed to the propagator U of the wave function, the propagator of the Wigner function and its Fourier transform κ will become independent of $\hbar$. Since the semiclassical Feynman propagator for the wave function is[16]

$$U(x'\mid x)=(2\pi\hbar)^{-1/2}\,|\partial_{12}S(x,x')|^{1/2}\exp\left[\frac{i}{\hbar}S(x,x')\right] \tag{14}$$

we obtain from Eqs. (7), (9), (12), and (13)

$$\kappa_{\mathrm{cl}}(x',\zeta'\mid x,\zeta)=(2\pi)^{-1}\,|\partial_{12}S(x,x')| \times\exp[-i\zeta\,\partial_1 S(x,x')-i\zeta'\,\partial_2 S(x,x')] \tag{15}$$

This propagator is actually the Fourier transform of the kernel of the Liouville operator, as it should be, since the classical limit of the Wigner function is a phase space probability density.

The lowest order quantum correction involves the terms listed in Eq. (13). Below we shall only concentrate on the cubic terms in the exponent, because it is due to them that the propagator will deviate from a singular conditional distribution and acquire a finite "width." The quadratic terms in the modulus are also qualitatively new with respect to the classical propagator. Their overall behavior, however, supports the conclusions we will draw from the study of the cubic exponent, so we omit here the discussion thereof.

In what follows we will establish the composition rule for the cubic coefficients based on the semigroup property of the propagator

$$\kappa''(x'',\zeta''\mid x,\zeta)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}dx'\,d\zeta'\,\kappa'(x'',\zeta''\mid x',\zeta')\,\kappa(x',\zeta'\mid x,\zeta) \tag{16}$$

We take the simplified form

$$\kappa(x', \zeta' \mid x, \zeta) \approx (2\pi)^{-1} |\partial_{12} S(x, x')| \times \exp\{-i[\zeta\, \partial_1 S(x, x') + \zeta'\, \partial_2 S(x, x')] - i\hbar^2[\zeta^3 \phi_{30}(x, x') + 3\zeta^2\zeta' \phi_{21}(x, x') + 3\zeta\zeta'^2 \phi_{12}(x, x') + \zeta'^3 \phi_{03}(x, x')]\} \tag{17}$$

and κ' similarly contains S' and ϕ'_{ij}. The question we would like to answer is what the cubic coefficients in the exponent of κ'' are, that is, how the transformation

$$\left\{\begin{matrix}\phi_{ij} \\ \phi'_{ij}\end{matrix}\right\} \to \phi''_{ij} \tag{18}$$

looks explicitly.

For $\hbar = 0$ the classical addition law for the actions is obtained,

$$S''(x, x'') = S(x, \bar{x}) + S'(\bar{x}, x'') \tag{19}$$

where the intermediate point $\bar{x}(x, x'')$ extremizes the sum as

$$\partial_2 S(x, \bar{x}) + \partial_1 S'(\bar{x}, x'') = 0 \tag{20}$$

In leading order in $\hbar$, small deviations from the extremizing point are allowed. Thus we expand the $O(\hbar^0)$ terms of the exponent linearly in $\varepsilon = x' - \bar{x}(x, x'')$, while we take the $O(\hbar^2)$ terms at $\varepsilon = 0$. Integration over ε results in a term proportional to the Dirac delta

$$\delta(\zeta' - (\partial_1 \bar{x})\zeta - (\partial_2 \bar{x})\zeta'') \tag{21}$$

which makes the integration by ζ' trivial. Finally, equating the coefficients of the respective cubic monomials in the exponents on both sides of Eq. (16), we obtain the transformation

$$\phi''_{30} = \phi_{30} + 3(\partial_1 \bar{x})\, \phi_{21} + 3(\partial_1 \bar{x})^2\, \phi_{12} + (\partial_1 \bar{x})^3\, (\phi_{03} + \phi'_{30}) \tag{22a}$$

$$\phi''_{21} = (\partial_2 \bar{x})\, \phi_{21} + 2(\partial_1 \bar{x})(\partial_2 \bar{x})\, \phi_{12} + (\partial_1 \bar{x})^2\, \phi'_{21} + (\partial_1 \bar{x})^2\, (\partial_2 \bar{x})(\phi_{03} + \phi'_{30}) \tag{22b}$$

$$\phi''_{12} = (\partial_1 \bar{x})\, \phi'_{12} + 2(\partial_1 \bar{x})(\partial_2 \bar{x})\, \phi'_{21} + (\partial_2 \bar{x})^2\, \phi_{12} + (\partial_1 \bar{x})(\partial_2 \bar{x})^2\, (\phi_{03} + \phi'_{30}) \tag{22c}$$

$$\phi''_{03} = \phi'_{03} + 3(\partial_2 \bar{x})\, \phi'_{12} + 3(\partial_2 \bar{x})^2\, \phi'_{21} + (\partial_2 \bar{x})^3\, (\phi_{03} + \phi'_{30}) \tag{22d}$$

For the sake of brevity we have omitted the functions' arguments, which one can retrace from

$$\begin{aligned}\phi_{ij} &= \phi_{ij}(x, \bar{x})\\ \phi'_{ij} &= \phi'_{ij}(\bar{x}, x'')\\ \phi''_{ij} &= \phi''_{ij}(x, x'')\\ \bar{x} &= \bar{x}(x, x'')\end{aligned} \tag{23}$$

4. QUANTUM PHASE CONDITION

The composition rule (22a)–(22d) bears a strong resemblance to the composition of covariance matrices in the case of classical noise.[6] There the matrix elements are analogous to the cubic coefficients ϕ_{ij} and ϕ'_{ij}, and the rule prescribes how two arbitrary initial covariance matrices conspire to produce a third one. The above composition law for quantal fluctuations does the same at the first glance. There is, however, an important restriction on the initial ϕ_{ij} and ϕ'_{ij} coefficients, which will turn out to be preserved along the time evolution. That restriction can be immediately seen by starting out from the semiclassical Feynman propagator in Eq. (14) and then constructing the propagator κ as prescribed by Eq. (7). Then the phase ϕ satisfies

$$\hbar\phi(x, \zeta, x', \zeta') = S\left(x + \frac{\hbar\zeta}{2}, x' + \frac{\hbar\zeta'}{2}\right) - S\left(x - \frac{\hbar\zeta}{2}, x' - \frac{\hbar\zeta'}{2}\right) \tag{24}$$

which results in the cubic coefficients

$$\phi_{ij}(x, x') = 1/24\, \partial_1^i \partial_2^j S(x, x') \tag{25}$$

Even if other corrections of $O(\hbar^2)$ are included in the Feynman propagator, the cubic coefficients remain those of Eq. (25).

It is a key obsrvation of this paper that relation (25) is invariant under time evolution of the propagator. Specifically, if ϕ_{ij} is determined by the classical action S through (25), and similar relation exists between a ϕ'_{ij} and an action S', then the resulting ϕ''_{ij} coefficients are given by Eq. (25), where the composed action S'' should replace S. We shall demonstrate that statement on the component ϕ''_{30}. According to Eq. (22a), the composition leads to

$$24\phi''_{30} = \partial_1^3 S + 3(\partial_1 \bar{x})\, \partial_1^2 \partial_2 S + 3(\partial_1 \bar{x})^2\, \partial_1 \partial_2^2 S + (\partial_1 \bar{x})^3\, (\partial_2^3 S + \partial_1^3 S') \tag{26}$$

where S and S' stand for $S(x, x')$ and $S'(\bar{x}, x'')$, respectively. On the other hand, by differentiating Eq. (20) twice in terms of x, one obtains the identity

$$(\partial_1 \bar{x})\, \partial_1^2 \partial_2 S + 2(\partial_1 \bar{x})^2\, \partial_1 \partial_2^2 S + (\partial_1 \bar{x})^3\, (\partial_2^3 S + \partial_1^3 S') = (\partial_1^2 \bar{x})\, \partial_{12} S \qquad (27)$$

Substitution of Eq. (27) into Eq. (26) results in

$$24\phi''_{30} = \partial_1^3 S + 2(\partial_1 \bar{x})\, \partial_1^2 \partial_2 S + (\partial_1 \bar{x})^2\, \partial_1 \partial_2^2 S + (\partial_1^2 \bar{x})\, \partial_{12} S \qquad (28)$$

The latter formula equals

$$\partial_1^3 S''(x, x'') \qquad (29)$$

as it can be seen by differentiating Eq. (19) thrice in terms of x and using Eq. (20). The derivation for the other components goes similarly. Thus we arrive at the important conclusion that in each iteration step of the time evolution of the propagator, the terms characterizing the quantum fluctuations in lowest order are determined through Eq. (25) by the action at the classical orbit. This is a general proof of the claim stated in ref. 10, based on one composition step in the case of the periodically kicked particle.

At this point a remark is in order. In Eq. (24) it is not necessary to restrict the action S to a time period T, as defined in Eq. (1). In principle the time interval can extend to arbitrarily long but finite times. Then relation (25) also holds between the actual S and ϕ_{ij}, and one always finds sufficiently small $\hbar$ so that the phase of the propagator κ becomes as in Eq. (17) with ϕ_{ij} substituted from Eq. (25). Our previous derivation can be considered, therefore, as a consistency proof within the path integral formalism. For long times it still remains an open question, however, how small should $\hbar$ be so that the approximation (17) with (25) remains valid. The problem of the appropriate rescaling of $\hbar$ will be discussed below.

The above results enable us to study quantum systems whose classical counterpart undergoes a Hamiltonian period-doubling scenario, or has a critical KAM torus with a noble winding number. The classical properties of such systems have been extensively discussed in the literature (see ref. 4 and references therein). For the sake of simplicity we will restrict ourselves to the period-doubling route; the derivation can be performed with little modification for the KAM problem.

The RNG transformation associated with the classical period doubling is defined on the action function as[17]

$$\hat{R}[S](x, x'') = \alpha\beta[S(x/\alpha, \bar{x}(x, x'')) + S(\bar{x}(x, x''), x''/\alpha)] \qquad (30)$$

Here the intermediate point $\bar{x}$ should extremize the right-hand side. The factors α and β are to be calculated from the fixed-point equation

$\hat{R}[S^*] = S^*$, which also determines the fixed-point action function S^* and the corresponding $\bar{x}^*$. Numerically, $\alpha = -4.018...$ and $\beta = 16.36...$ are obtained.[5,17] The fixed-point function is universal in the sense that it attracts the action function of any area-preserving map at the accumulation point of its period-doubling scenario. The factor $\delta_H = 8.721...$, which scales the control parameter and is the Hamiltonian equivalent of Feigenbaum's δ, is obtained as the dominant eigenvalue of the linearized RNG transformation around the fixed point.[5,17]

We wish to apply the RNG transformation to the quantum fluctuations in classically period-doubling systems. Therefore the transformation is to be extended to the propagator κ. Using the composition law (16), we obtain

$$\hat{R}[\kappa](x'', \zeta'' \mid x, \zeta) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dx'\, d\zeta' \times \kappa(x''/\alpha, \beta\zeta'' \mid x', \zeta')\, \kappa(x', \zeta' \mid x/\alpha, \beta\zeta) \tag{31}$$

This transformation reduces to Eq. (30) in the limit $\hbar \to 0$, if the propagator is taken in the form of Eq. (17). Assume now that the classical action appearing in Eq. (17) is the fixed-point function S^*. In this case the composition rules (22a)–(22d) define the linearized RNG transformation $\hat{R}[\phi]$ on the cubic coefficients ϕ_{ij} as follows. Let both ϕ_{ij} and ϕ'_{ij} in Eqs. (22a)–(22d) be identical and the intermediate point be the universal $\bar{x}^*$. Then

$$\hat{R}[\phi]_{ij}\,(x, x'') = \beta^3 \phi''_{ij}(x/\alpha, x''/\alpha) \tag{32}$$

Instead of a general spectral analysis of the linear RNG transformation (32), we must rather consider the set of coefficients ϕ''_{ij} related to the fixed-point action S^* via Eq. (25). It is this set ϕ''_{ij} to which the initial coefficients ϕ_{ij} converge if the classical action S converges to S^*. Our main observation is that ϕ''_{ij} forms an eigenvector of the RNG transformation,

$$\hat{R}_\phi[\phi^*]_{ij} = \lambda \phi^*_{ij} \tag{33}$$

where the eigenvalue is

$$\lambda = \alpha^2 \beta^2 \tag{34}$$

as one can easily convince oneself via elementary algebra. Since the coefficients ϕ_{ij} characterize the $O(\hbar^2)$ term, Planck's constant scales with $|\alpha\beta|$, in accordance with refs. 9–12.

5. DISCUSSION

An alternative derivation of the quantum phase condition, Eq. (25), sheds further light on the difference between classical and quantum noise. Consider again Eq. (7), which can be regarded as a *factorization* of the Wigner propagator κ into a pair of quantum propagators U. *Exact* quantum mechanics *preserves* this factorization. Therefore, an *exact* calculation of Eq. (16) expresses κ'' as a pair of propagators U'', where

$$U''(x'' \mid x) = \int_{-\infty}^{\infty} dx'\, U'(x'' \mid x')\, U(x' \mid x) \tag{35}$$

This may be seen by using integration variables $x' \pm \hbar\zeta'/2$ in Eq. (16).

However, the semiclassical approximation *also* preserves the factorization. What is more, it is well known that if (35) is calculated in semiclassical approximation, i.e., using the form (14) for U and the stationary phase approximation for the integral, then the semiclassical approximation to the propagator U'' is given by Eq. (14) using the action S'' of Eq. (19). Thus Eq. (25) is invariant in time.

One curiosity is the following. In doing Eq. (35) by stationary phase the exponent

$$S'(x'', x') + S(x', x) \tag{36}$$

is expanded in x' to second order about the stationary phase point $\bar{x}$, given by (20). In other words, the approximate exponent involves at most second derivatives of the S's with respect to their arguments. On the other hand, in doing the integral of Eq. (16) by the method of Section 3, third derivatives of the actions S are encountered.

If it were indeed true that the standard stationary phase calculation on (35) led to third-order errors in the phase S'', then the calculation of Section 3 would actually be going beyond stationary phase. However, it is clear that one could have kept the expansion of (36) to third order in $(x' - \bar{x})$ without introducing corrections to the phase. Therefore, the phase calculation (19) is actually correct to third order, and the time independence of (25) follows.

The factorization of the Wigner propagator just discussed seems to be an essentially quantum symmetry. It is not possible that classical random noise preserves a factorization, even assuming one to be present in a classical propagator for the probability density. Another fact should be emphasized, namely that random noise has infinitely many degrees of freedom, while the nonvanishing quantum cumulants are determined by the classical action.

It remains to study whether the RNG analysis can be extended to higher orders in $\hbar$, within the framework of corrections to the quasiclassical approximation. Jensen and Niu,[18] by a totally different approach, found numerically that the next order, $\hbar^4$, scaled in a way consistent with $(\alpha\beta)^4$.

In conclusion, we have generalized earlier renormalization approaches to quantum noise for systems which classically exhibit critical behavior. The framework shows a strong analogy with the renormalization method for classical noise[6]; however, a specifically quantum requirement introduces a restriction: The "widening" of the propagator of the Wigner function, in lowest order in $\hbar$, is directly determined by the classical action. That property proves to be a symmetry of the renormalization transformation; thus, it is preserved along physically meaningful renormalization trajectories. Near the fixed point the quantum fluctuations are characterized by an eigenvalue that can be written in terms of classical scale factors.

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